*-Ideals and Formal Morita Equivalence of *-Algebras

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Abstract

Motivated by deformation quantization, we introduced in an earlier work the notion of formal Morita equivalence in the category of *-algebras over a ring C which is the quadratic extension by i of an ordered ring R. The goal of the present paper is twofold. First, we clarify the relationship between formal Morita equivalence, Ara's notion of Morita *-equivalence of rings with involution, and strong Morita equivalence of C^*-algebras. Second, in the general setting of *-algebras over C, we define 'closed' *-ideals as the ones occuring as kernels of *-representations of these algebras on pre-Hilbert spaces. These ideals form a lattice which we show is invariant under formal Morita equivalence. This result, when applied to Pedersen ideals of C^*-algebras, recovers the so-called Rieffel correspondence theorem. The triviality of the minimal element in the lattice of closed ideals, called the 'minimal ideal', is also a formal Morita invariant and this fact can be used to describe a large class of examples of *-algebras over C with equivalent representation theory but which are not formally Morita equivalent. We finally compute the closed *-ideals of some *-algebras arising in differential geometry.

Contents

1 Introduction

The concept of formal deformation quantization, first introduced in [3], has been used successfully to construct and classify the quantum observable algebras of physical systems in terms of classical data, generally given by a symplectic or Poisson manifold (see [4, 11, 13, 17, 22–24, 34] for existence and classifications and [14, 31, 33] for recent reviews). One is naturally led to investigate representations of these algebras, as different representations can be used to describe different physical situations. It is then important to decide what kind of representations (or category of modules) of the observable algebras have physical meaning and should be considered. It turns out that the usual *-representation theory of C^*-algebras provides a guide for the formulation of representations of quantum observable algebras arising in deformation quantization. In particular, one can always consider star-products compatible with the natural *-involution on functions (given by complex conjugation), and we note that the underlying ring of real formal power series $\mathbb{R}[[\hbar]]$ has a natural order structure. Combining these two facts, one can define positive $\mathbb{C}[[\hbar]]$-linear functionals on the star-product algebras and *-representations on pre-Hilbert spaces over $\mathbb{C}[[\hbar]]$ analogous to the C^*-algebra case. For example, every positive linear functional gives rise to a GNS construction of a *-representation (see [8] for a detailed exposition of these ideas). These notions have yielded physically interesting representations and, in fact, most known operator representations are of this type, see e.g. [5–7, 32].

With this motivation, we started in [9, 10] a more systematic study of *-algebras over $\mathsf{C} = \mathsf{R}(i)$, where R is an arbitrary (commutative, unital) ordered ring and C is its quadratic extension by i, with $i^2 = -1$, and their representations on pre-Hilbert spaces over C. We note that not only star-product algebras over $\mathbb{C}[[\hbar]]$ but also arbitrary *-algebras over $\mathbb{C}$ (such as *-algebras of unbounded operators, see e.g. [30]) fit into this framework. In this setting, one can consider a purely algebraic version of Rieffel induction and define the notion of formal Morita equivalence [9], analogous to C^*-algebras [18, 27, 28]. In the present paper, we continue the investigation of *-representations of *-algebras and formal Morita equivalence.

First, we clarify the relationship between formal Morita equivalence, Ara's notion of Morita *-equivalence of rings with involution [1, 2] and strong Morita equivalence of C^*-algebras [29]. For non-degenerate and idempotent *-algebras over C, we show that formal Morita equivalence implies Morita *-equivalence. In the case where the *-algebras are given by Pedersen ideals of C^*-algebras [25], we show that, in fact, formal Morita equivalence and Morita *-equivalence are equivalent. It then follows immediately from Ara's recent algebraic characterization of strong Morita equivalence of C^*-algebras in terms of their Pedersen ideals [1, Thm. 2.4] that two C^*-algebras are strongly Morita equivalent if and only if their Pedersen ideals are formally Morita equivalent.

Second, we discuss *-ideals in *-algebras over C, their behavior under formal Morita equivalence and how they can be used to describe particular properties of star-product algebras which general *-algebras over C may lack. In a *-algebra over C, we define an ideal to be 'closed' if it occurs as the kernel of some *-representation on a pre -Hilbert space over C. These ideals form a lattice, which we show is invariant under formal Morita equivalence. In the case of C^*-algebras, an ideal is closed in this algebraic sense if and only if it is norm closed and the aforementioned invariance reduces to the so-called Rieffel correspondence theorem (see [26, Prop. 3.24]). We note that *-algebras having a "large" number of positive linear functionals possess nice features which do not hold in general. For instance, they admit faithful representations on pre-Hilbert spaces (see [9, Prop. 2.8]). We observe that not only C^*-algebras but also star-product algebras arising in deformation quantization have this property [10, Prop. 5.3]. Admitting a faithful representation is equivalent to the triviality of the minimal closed ideal (defined as the intersection of all closed *-ideals), which we also show is a formal Morita invariant. We use this invariant to describe a large class of examples of *-algebras having equivalent categories of (strongly non-degenerate) *-representations on pre-Hilbert spaces but which are not formally Morita equivalent, generalizing [9, Cor. 5.20]. (It was shown in [9, Thm. 5.10] that formal Morita equivalence implies the equivalence of categories of strongly non-degenerate *-representations.)

Finally, we apply these general algebraic notions to the particular case of the *-algebra $C^\infty(M)$ of complex-valued smooth functions on a manifold M. We show that closed *-ideals in this example are given by the ideals of functions vanishing on closed subsets of M, and hence agree with the closed ideals with respect the (weak) C^0-topology ([35, Thm. 3]). We also show that $C^\infty(M)$ is formally Morita equivalent to the *-algebra of smooth sections of the endomorphism bundle of any Hermitian vector bundle E over M. These facts are not surprising since their analogues in the continuous (C^*-algebraic) category (in the sense of strong Morita equivalence) are well-known. They are, however, conceptually important since they show that *-algebras in the smooth category can be treated independently through general algebraic methods (we note that M is *not* assumed to be compact and hence the *-algebras in questions are *not* pre-C^*-algebras). We point out that this discussion is the 'classical' starting point in the investigation of these ideas in the framework of formal deformation quantization, see also [19] for the case of a trivial bundle.

The paper is organized as follows. In Section 2, we recall the notions of ordered rings, *-algebras, and their *-representations, including the GNS construction (details can be found in [8, 9]). In Section 3, we describe how the notions of formal Morita equivalence, Morita *-equivalence and strong Morita equivalence are related. We define closed *-ideals in Section 4 and discuss basic properties of the minimal closed *-ideal and its relation to GNS representations and positive functionals. In Section 5, we focus on algebraic Rieffel induction and formal Morita equivalence of *-algebras and prove that the lattice of closed *-ideals is a formal Morita invariant. Section 6 contains the description of closed *-ideals of $C^\infty(M)$. As a *-algebra with non-trivial minimal ideal we consider the sections of the Grassmann bundle over M. We also show in this section that for any Hermitian vector bundle E over a manifold M, $C^\infty(M)$ and $\Gamma^\infty(\mathsf{End}(E))$ are formally Morita equivalent, as well as $C_0^\infty(M)$ and $\Gamma_0^\infty(\mathsf{End}(E))$.

2   Preliminaries

In this section, we just recall some basic facts on ordered rings, pre-Hilbert spaces and *-algebras to establish our notation (see [8, 9] for details).

Let R be an associative commutative ring with $1 \neq 0$ which is *ordered*, i.e. equipped with a distinguished subset P, the positive elements, such that R is the disjoint union $-\mathsf{P} \cup \{0\} \cup \mathsf{P}$ and one has $\mathsf{P} + \mathsf{P}, \mathsf{P} \cdot \mathsf{P} \subseteq \mathsf{P}$. We write $a > b$ if $a - b \in \mathsf{P}$ etc. In particular $1 > 0$, and R has

characteristic zero and no zero divisors. Next we consider the canonical quadratic ring extension $\mathsf{C} = \mathsf{R}(\mathrm{i})$ where $\mathrm{i}^2 = -1$. Then C becomes an associative commutative ring with $1 \neq 0$ which has again characteristic zero and no zero divisors. Complex conjugation $z \mapsto \overline{z}$ is defined as usual for $z \in \mathsf{C}$ and the embedding $\mathsf{R} \hookrightarrow \mathsf{C}$ determines the real elements of C, i.e. those with $z = \overline{z}$. Finally, note that $\overline{z}z \geq 0$ and $\overline{z}z = 0$ if and only if $z = 0$. Examples of ordered rings are of course $\mathbb{Z}$, $\mathbb{R}$, and the formal power series $\mathbb{R}[[\lambda]]$, which is the important example for formal deformation quantization.

A *pre-Hilbert space* over C is a C-module $\mathfrak{H}$ with positive-definite Hermitian product $\langle \cdot, \cdot \rangle$: $\mathfrak{H} \times \mathfrak{H} \to \mathsf{C}$ which is linear in the second argument (physicists' convention). We immediately have the Cauchy-Schwarz inequality $\langle \phi, \psi \rangle \langle \psi, \phi \rangle \leq \langle \phi, \phi \rangle \langle \psi, \psi \rangle$ for all $\phi, \psi \in \mathfrak{H}$. For a pre-Hilbert space $\mathfrak{H}$, we say that $A \in \mathsf{End}_{\mathsf{C}}(\mathfrak{H})$ has a (necessarily unique) adjoint $A^* \in \mathsf{End}_{\mathsf{C}}(\mathfrak{H})$ if for all $\phi, \psi \in \mathfrak{H}$ the equality $\langle A^*\phi, \psi \rangle = \langle \phi, A\psi \rangle$ holds. One defines $\mathfrak{B}(\mathfrak{H}) := \{A \in \mathsf{End}_{\mathsf{C}} \mid A^* \text{ exists } \}$ and easily finds that $\mathfrak{B}(\mathfrak{H})$ is an associative algebra over C with unit element $\mathsf{id}_{\mathfrak{H}}$ and a $*$-involution $A \mapsto A^*$. More generally, one defines a $*$-*algebra* $\mathcal{A}$ to be an associative algebra over C equipped with a $*$-involution, i.e. an antilinear involutive anti-automorphism. Hermitian, normal, isometric and unitary elements in $\mathcal{A}$ are defined as usual. An *approximate identity* consists of a filtration $\mathcal{A} = \bigcup_{\alpha \in I} \mathcal{A}_\alpha$ by C-submodules $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ indexed by a directed set I and elements $\{E_\alpha\}_{\alpha \in I}$ such that $E_\alpha^* = E_\alpha$, $E_\alpha E_\beta = E_\alpha = E_\beta E_\alpha$ for $\alpha < \beta$ and $E_\alpha A = A = A E_\alpha$ for all $A \in \mathcal{A}_\alpha$ and $\alpha \in I$. A linear functional $\omega : \mathcal{A} \to \mathsf{C}$ is called <u>positive</u> if $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. Then one has the Cauchy-Schwarz <u>inequality</u> $\omega(A^*B)\overline{\omega(A^*B)} \leq \omega(A^*A)\omega(B^*B)$ as well as the 'almost reality' property $\omega(A^*B) = \overline{\omega(B^*A)}$, which implies reality $\omega(A^*) = \overline{\omega(A)}$ if e.g. $\mathcal{A}$ has an approximate identity. Using positive functionals, we are able to define positive algebra elements as well: by $\mathcal{A}^{++}$ we denote the *algebraically positive elements* of the form $A = a_1 A_1^* A_1 + \cdots + a_n A_n^* A_n$ where $a_i > 0$ and $A_i \in \mathcal{A}$. By $\mathcal{A}^+$ we denote the *positive* elements of $\mathcal{A}$, where $A \in \mathcal{A}^+$ if A is Hermitian and $\omega(A) \geq 0$ for all positive linear functionals $\omega : \mathcal{A} \to \mathsf{C}$. We note that if $\mathcal{A}$ is a C^*-algebra, then these notions of positivity agree with the standard ones. Moreover, in this case $\mathcal{A}^+ = \mathcal{A}^{++}$. These notions also yield the expected results for $C^\infty(M)$, see [9, App. B].

A $*$-*representation* of a $*$-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathfrak{H}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathfrak{B}(\mathfrak{H})$. Such a $*$-representation is called *non-degenerate* if $\pi(A)\phi = 0$ for all $A \in \mathcal{A}$ implies $\phi = 0$ and *strongly non-degenerate* if the C-span of all $\pi(A)\phi$ with $A \in \mathcal{A}$ and $\phi \in \mathfrak{H}$ coincides with $\mathfrak{H}$. In general, strong non-degeneracy implies non-degeneracy and if $\mathcal{A}$ has a unit both notions agree and are equivalent to the condition $\pi(1) = \mathsf{id}_{\mathfrak{H}}$.

If $\omega : \mathcal{A} \to \mathsf{C}$ is a positive linear functional, then $\mathcal{J}_\omega := \{A \in \mathcal{A} \mid \omega(A^*A) = 0\}$ is a left ideal in $\mathcal{A}$, the so-called Gel'fand ideal and hence $\mathfrak{H}_\omega := \mathcal{A}/\mathcal{J}_\omega$ is a $\mathcal{A}$-left module. Moreover, $\mathfrak{H}_\omega$ becomes a pre-Hilbert space via $\langle \psi_B, \psi_C \rangle := \omega(B^*C)$ such that the left action $\pi_\omega(A)\psi_B := \psi_{AB}$ is a $*$-representation, the so-called GNS representation. Here $\psi_B \in \mathfrak{H}_\omega$ denotes the equivalence class of $B \in \mathcal{A}$. This GNS construction will play a crucial role in this paper (see [8] for an extensive treatment).

Let us recall the notion of algebraic Rieffel induction, as discussed in [9]. Let $\mathcal{A}$ be a $*$-algebra over C and let $\mathfrak{X}$ be a right $\mathcal{A}$-module (*all* modules over $*$-algebras considered here will be assumed to have a compatible C-module structure). An $\mathcal{A}$-*valued inner product* is a map $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathcal{A}$, which is C-linear and $\mathcal{A}$-right linear in the second argument and satisfies $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$, for all $x, y \in \mathfrak{X}$. We call this inner product *positive semi-definite* if $\langle x, x \rangle_{\mathcal{A}} \in \mathcal{A}^+$ and *positive definite* if in addition $\langle x, x \rangle_{\mathcal{A}} = 0$ implies $x = 0$. We say it is *full* if the C-span of all $\langle x, y \rangle_{\mathcal{A}}$ is $\mathcal{A}$. A (full, positive semi-definite) $\mathcal{A}$-valued inner product on a left $\mathcal{A}$-module is defined analogously, but with linearity properties (with respect to C and $\mathcal{A}$) in the first argument. If $\mathcal{B}$ is another $*$-algebra over C, and ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule with an $\mathcal{A}$-valued inner product, we call the $\mathcal{B}$-action *compatible* if $\langle B \cdot x, y \rangle_{\mathcal{A}} = \langle x, B^* \cdot y \rangle_{\mathcal{A}}$ for all $B \in \mathcal{B}$ and $x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$.

Let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule with positive semi-definite $\mathcal{A}$-valued inner product and com-

patible $\mathcal{B}$-action. If $\pi_{\mathcal{A}}$ is a *-representation of $\mathcal{A}$ on $\mathfrak{H}$, then one can consider the C-module $\widetilde{\mathfrak{K}} := {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}$, where the $\mathcal{A}$-balanced tensor product is defined using $\pi_{\mathcal{A}}$. Then $\widetilde{\mathfrak{K}}$ is a $\mathcal{B}$-left module and by $\langle x \otimes \psi, y \otimes \phi \rangle_{\widetilde{\mathfrak{K}}} := \langle \psi, \pi_{\mathcal{A}}(\langle x, y \rangle_{\mathcal{A}})\phi \rangle_{\mathfrak{H}}$ one obtains a Hermitian inner product on $\widetilde{\mathfrak{K}}$. We say that the bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies the property $\mathbf{P}$ if this Hermitian product is positive semi-definite for all *-representations $(\pi_{\mathcal{A}}, \mathfrak{H})$ of $\mathcal{A}$, see [9, Sect. 3 and 4] for various sufficient conditions guaranteeing $\mathbf{P}$. In this case one can quotient out the length zero vectors $\widetilde{\mathfrak{K}}^{\perp}$ to obtain a pre-Hilbert space $\mathfrak{K} = \widetilde{\mathfrak{K}}/\widetilde{\mathfrak{K}}^{\perp}$. It is easy to check that the left action of $\mathcal{B}$ passes to the quotient and yields a *-representation $\pi_{\mathcal{B}}$ on $\mathfrak{K}$. The whole procedure, called *algebraic Rieffel induction*, is functorial and the functor is denoted by $\mathfrak{R}_{\mathfrak{X}}$. Given a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule with a positive semi-definite $\mathcal{B}$-valued inner product and compatible $\mathcal{A}$-action, we can consider the corresponding conjugated $(\mathcal{A}\text{-}\mathcal{B})$-bimodule, denoted by $\overline{\mathfrak{X}}$, which is just $\mathfrak{X}$ as additive group, but with $\mathcal{A}$ and $\mathcal{B}$ acting by adjoints and C by complex conjugates. We then say that ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ satisfies property $\mathbf{Q}$ if $\overline{\mathfrak{X}}$ satisfies $\mathbf{P}$ as above (see [9, Sect. 5]).

Let us now consider formal Morita equivalence, as in [9]. Suppose ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule, equipped with full positive semi-definite $\mathcal{A}$-valued and $\mathcal{B}$-valued inner products, compatible $\mathcal{A}$ and $\mathcal{B}$ actions and so that properties $\mathbf{P}$ and $\mathbf{Q}$ are satisfied. We also assume that the inner products are *compatible*, in the sense that $x \cdot \langle y, z \rangle_{\mathcal{A}} = {}_{\mathcal{B}}\langle x, y \rangle \cdot z$ for all $x, y, z \in \mathfrak{X}$. We call this object an *equivalence bimodule* and say that two *-algebras $\mathcal{A}$ and $\mathcal{B}$ are *formally Morita equivalent* if there exists a $(\mathcal{B}\text{-}\mathcal{A})$-equivalence bimodule. We recall that if the actions of $\mathcal{A}$ and $\mathcal{B}$ on ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ are in addition strongly non-degenerate, then ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ is called a *non-degenerate equivalence bimodule* and one can show that in this case that $\mathcal{A}$ and $\mathcal{B}$ have equivalent categories of (strongly non-degenerate) *-representations, with the functors $\mathfrak{R}_{\mathfrak{X}}$ and $\mathfrak{R}_{\overline{\mathfrak{X}}}$ defining this equivalence ([9, Thm. 5.10]). As for notation, we will write *-$\mathsf{rep}(\mathcal{A})$ for the category of *-representations of $\mathcal{A}$ (usually considered with isometric interwiners as morphisms) and *-$\mathsf{Rep}(\mathcal{A})$ for the category of strongly non-denerate *-representations.

A *-algebra over C has *sufficiently many positive linear functionals* if for every non-zero Hermitian element H one finds a positive linear functional ω with $\omega(H) \neq 0$. We recall the following important fact: a *-algebra $\mathcal{A}$ *with approximate identity* has sufficiently many positive linear functionals if and only if $\mathcal{A}$ has a faithful *-representation. As a consequence, these algebras present some nice properties, resembling C^*-algebras. For instance, they are torsion-free ($zA = 0$ implies $A = 0$ or $z = 0$, for $z \in \mathsf{C}$), $A^*A = 0$ implies $A = 0$ and $H^n = 0$ for a normal element H implies $H = 0$ ([9, Prop. 2.8]). We observe that the assumption on the existence of an approximate identity can be weakened in particular cases. Indeed, if $\mathcal{A}$ is a C^*-algebra or a star-product algebra, then it always has sufficiently many positive linear functionals but it does not admit an algebraic approximate identity in general. Nevertheless, these algebras always have faithful representations and all the aforementioned nice properties. We observe that in both cases, however, a topological approximate identity exists (with respect to the norm topology in the case of C^*-algebras and with respect to the λ-adic topology for star-product algebras, see [9, Sect. 8]).

On the other hand, the existence of sufficiently many positive linear functionals, without any further requirement, is generally not enough to guarantee the existence of faithful representations and desirable algebraic properties. For instance, consider $\mathsf{C} = \mathsf{R}(i)$ with its natural C-module structure. Define the *-algebra $\mathcal{A}$ to be this C-module, with the natural *-involution but with zero multiplication. Then *all* linear functionals on $\mathcal{A}$ are positive and $\mathcal{A}$ has sufficiently many positive linear functionals. Note that $\mathcal{A}$ does not admit faithful representations (since we have $A \neq 0$ with $A^*A = 0$). For an example of a *-algebra *without* sufficiently many positive linear functionals, let $\bigwedge{}^{\bullet}(\mathsf{C}^n)$ be the Grassmann algebra over C^n, equipped with a *-involution given by $e_i^* := e_i$, where $e_1, \dots, e_n$ is the canonical basis of C^n. Denote by $\bigwedge{}^{+}(\mathsf{C}^n)$ the ideal of elements with positive degree and write $\bigwedge{}^{\bullet}(\mathsf{C}^n) = \mathsf{C}1 \oplus \bigwedge{}^{+}(\mathsf{C}^n)$. Then every positive linear functional $\omega : \bigwedge{}^{\bullet}(\mathsf{C}^n) \to \mathsf{C}$

has to satisfy $\omega(1) \geq 0$ and $\omega|_{\bigwedge^+(\mathsf{C}^n)} = 0$. Hence, up to normalization, there is only one positive functional. In particular, every (Hermitian) element in $\bigwedge^+(\mathsf{C}^n)$ is positive but $\bigwedge^\bullet(\mathsf{C}^n)$ does not have sufficiently many positive linear functionals for $n \geq 1$. Note that this *-algebra is nevertheless torsion-free. For an example with torsion, set $\mathsf{R} = \mathbb{Z}$ and $\mathsf{C} = \mathbb{Z} \oplus i\mathbb{Z}$ and consider $\mathbb{Z}_2 = \{0, 1\}$. Then $\mathbb{Z}_2$ becomes a unital *-algebra over C by $i \cdot 1 = 1$ and $1^* = 1$. Note that $\mathbb{Z}_2$ is not torsion-free, since for instance $2 \cdot 1 = 0$, and there are no C-linear functionals $\omega : \mathbb{Z}_2 \to \mathsf{C}$ except for the zero functional.

Thus the general category of *-algebras over C seems too wide for the interpretation of these algebras as 'observable algebras'. We will discuss in Sections 4 and 5 how the ideal strucuture of these algebras help characterizing the more 'observable algebra'-like objects.

3 Formal Morita equivalence and C^*-algebras

The aim of this section is to clarify the relationship between the notions of formal Morita equivalence [9], Morita *-equivalence [2] and strong Morita equivalence [26, 29]. We start by briefly recalling some of the definitions.

Let $\mathcal{A}$ be a C^*-algebra. A *right pre-Hilbert $\mathcal{A}$-module* is a right $\mathcal{A}$-module $\mathfrak{X}$, equipped with a positive definite $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ We call $\mathfrak{X}_{\mathcal{A}}$ a *right Hilbert $\mathcal{A}$-module* if $\mathfrak{X}$ is in addition complete under the norm $\|x\|_{\mathcal{A}} = \| \langle x, x \rangle_{\mathcal{A}}^{1/2} \|$. A *left Hilbert $\mathcal{A}$-module* is defined analogously. A Hilbert $\mathcal{A}$-module is called *(topologically) full* if the ideal $\langle \mathfrak{X}, \mathfrak{X} \rangle_{\mathcal{A}}$ is dense in $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then an *imprimitivity bimodule* is a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ which is a full right $\mathcal{A}$-Hilbert module and a full left $\mathcal{B}$-Hilbert module so that for all $x, y, z \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and $A \in \mathcal{A}$, $B \in \mathcal{B}$, we have $\langle B \cdot x, y \rangle_{\mathcal{A}} = \langle x, B^* \cdot y \rangle_{\mathcal{A}}$, ${}_{\mathcal{B}}\langle x \cdot A, y \rangle = {}_{\mathcal{B}}\langle x, y \cdot A^* \rangle$ and $x \cdot \langle y, z \rangle_{\mathcal{A}} = {}_{\mathcal{B}}\langle x, y \rangle \cdot z$. We finally say that two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are *strongly Morita equivalent* if there is a $(\mathcal{B}\text{-}\mathcal{A})$-imprimitivity bimodule. An equivalent definition of strong Morita equivalence can be given as follows: $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent if there exists a full right $\mathcal{A}$-Hilbert module $\mathfrak{X}_{\mathcal{A}}$ so that $\mathcal{B} \cong \mathcal{K}(\mathfrak{X}_{\mathcal{A}})$, where $\mathcal{K}(\mathfrak{X}_{\mathcal{A}})$ is the completion of the *-algebra $\mathfrak{F}(\mathfrak{X}_{\mathcal{A}})$ of "finite-rank operators" on $\mathfrak{X}_{\mathcal{A}}$, that is, the linear span of operators of the form $\Theta_{x,y}$, where $\Theta_{x,y} \cdot z = x \cdot \langle y, z \rangle_{\mathcal{A}}$, for $x, y, z \in \mathfrak{X}_{\mathcal{A}}$.

To state Ara's result on the algebraic characterization of strong Morita equivalence, we will need the following definitions. Recall that a ring R is called *non-degenerate* if $r \cdot R = 0$ or $R \cdot r = 0$ implies that $r = 0$ for all $r \in R$, and *idempotent* if elements of the form $r_1 r_2$ span R. We also consider the category $\mathsf{Mod}\text{-}R$ of right R-modules $\mathcal{E}$ satisfying $\mathcal{E} \cdot R = \mathcal{E}$ (in our terminology, this means that the action of R on $\mathcal{E}$ is strongly non-degenerate) and $x \cdot R = 0$ implies $x = 0$ for all $x \in \mathcal{E}$, i.e. the R-action on $\mathcal{E}$ is non-degenerate. Let now R and S be two non-degenerate and idempotent rings with involution. Then R and S are called *Morita *-equivalent* if there exists a right R-module $\mathcal{E} \in \mathsf{Mod}\text{-}R$ equipped with a full R-valued inner product $\langle \cdot, \cdot \rangle$ so that $\langle x, \mathcal{E} \rangle = 0$ implies $x = 0$ for $x, y \in \mathcal{E}$ and $S \cong \mathfrak{F}(\mathcal{E}_R)$. Here $\mathfrak{F}(\mathcal{E}_R)$ again denotes the *-algebra of "finite rank" operators on $\mathcal{E}_R$ and is defined as before for C^*-algebras. We remark that if R is a K-algebra, for some fixed unital and commutative ring K, and $\mathcal{E}$ has a compatible K-module structure, then it follows from the non-degeneracy of R that $\langle \cdot, \cdot \rangle$ is automatically K-linear in the second argument.

With this notion of Morita *-equivalence, one can give a purely algebraic characterization of strong Morita equivalence of C^*-algebras. Recall that if $\mathcal{A}$ is a C^*-algebra, then its *Pedersen ideal* $\mathcal{P}_{\mathcal{A}}$ is defined as the minimal dense ideal of $\mathcal{A}$, see [25, Sect. 5.6]. One has then the following result, due to Ara ([1, Thm. 2.4]): *Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent if and only if $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ are Morita *-equivalent.* We will now discuss how this result is related to the notion of formal Morita equivalence. We start with the following observation.

Lemma 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be non-degenerate, idempotent *-algebras over $\mathbb{C}$. Then if $\mathcal{A}$ and $\mathcal{B}$ are formally Morita equivalent, they are also Morita *-equivalent.*

PROOF: Note that it suffices to show that there exists an equivalence bimodule $_\mathcal{B}\mathfrak{X}_\mathcal{A}$ so that $\langle x, \mathfrak{X}\rangle_\mathcal{A} = 0$ implies $x = 0$, for $x \in \mathfrak{X}$, $(\mathfrak{X}_\mathcal{A}, \langle \cdot, \cdot\rangle_\mathcal{A}) \in \mathsf{Mod}\text{-}\mathcal{A}$ (i.e., the $\mathcal{A}$-action on $\mathfrak{X}$ is strongly non-degenerate and $x \cdot \mathcal{A} = 0$ implies $x = 0$ for $x \in \mathfrak{X}$) and $\mathcal{B} \cong \mathfrak{F}(\mathfrak{X}_\mathcal{A})$.

Let $_\mathcal{B}\mathfrak{X}_\mathcal{A}$ be an arbitrary equivalence bimodule. We start by noticing that we can choose it so that the $\mathcal{B}$ and $\mathcal{A}$ actions are strongly non-degenerate. Indeed, it follows from the compatibility of $\langle \cdot, \cdot\rangle_\mathcal{A}$ and $_\mathcal{B}\langle \cdot, \cdot\rangle$ and their fullness properties that $\mathfrak{X} \cdot \mathcal{A} = \mathcal{B} \cdot \mathfrak{X}$. If we denote $\hat{\mathfrak{X}} = \mathfrak{X} \cdot \mathcal{A} = \mathcal{B} \cdot \mathfrak{X}$, then $\hat{\mathfrak{X}}$ has a natural $(\mathcal{B}\text{-}\mathcal{A})$-bimodule structure and also compatible $\mathcal{A}$ and $\mathcal{B}$ valued inner products, defined by simply restricting $_\mathcal{B}\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle_\mathcal{A}$ to $\hat{\mathfrak{X}}$. Note that given $A \in \mathcal{A}$, by idempotency we can write $A = \sum_i A_i^1 A_i^2 A_i^3$, for $A_i^j \in \mathcal{A}$, and by fullness of $\langle \cdot, \cdot\rangle_\mathcal{A}$, we can write $A_i^2 = \sum_j \langle x_j^i, y_j^i\rangle_\mathcal{A}$. So $A = \sum_i A_i^1 \sum_j \langle x_j^i, y_j^i\rangle_\mathcal{A} A_i^3 = \sum_{i,j} \langle x_j^i {A_i^1}^*, y_j^i A_i^3\rangle_\mathcal{A}$, showing that $\langle \cdot, \cdot\rangle_\mathcal{A}$ restricted to $\hat{\mathfrak{X}}$ is still full, and the same holds for $\langle \cdot, \cdot\rangle_\mathcal{B}$. Therefore $\hat{\mathfrak{X}}$ is a $(\mathcal{B}\text{-}\mathcal{A})$-equivalence bimodule with strongly non-degenerate $\mathcal{A}$ and $\mathcal{B}$ actions.

We also observe that if $_\mathcal{B}\mathfrak{X}_\mathcal{A}$ is an (arbitrary) equivalence bimodule, then $\mathcal{B}$ and $\mathcal{A}$ act on $\mathfrak{X}$ faithfully (i.e., $B \cdot x = 0$ for all $x \in \mathfrak{X}$ implies $B = 0$, for all $B \in \mathcal{B}$, and the same for $\mathcal{A}$). To see that, suppose $B \cdot x = 0$ for all x. Then given an arbitrary $B' \in \mathcal{B}$, it follows that we can write $B' = \sum_i {}_\mathcal{B}\langle x_i, y_i\rangle$ and hence $B \cdot B' = B\sum_i {}_\mathcal{B}\langle x_i, y_i\rangle = \sum_i {}_\mathcal{B}\langle Bx_i, y_i\rangle = 0$. Therefore $B = 0$ by non-degeneracy of $\mathcal{B}$ and the same argument holds for $\mathcal{A}$. It then follows (see the proof of [9, Prop. 5.16]) that $\{x \in {}_\mathcal{B}\mathfrak{X}_\mathcal{A} \mid \langle x, y\rangle_\mathcal{A} = 0, \forall y \in {}_\mathcal{B}\mathfrak{X}_\mathcal{A}\} = \{x \in {}_\mathcal{B}\mathfrak{X}_\mathcal{A} \mid {}_\mathcal{B}\langle x, y\rangle = 0, \forall y \in {}_\mathcal{B}\mathfrak{X}_\mathcal{A}\} = N$ and $\mathfrak{X}/N$ is still a $(\mathcal{B}\text{-}\mathcal{A})$-equivalence bimodule, with the property that $\langle x, \mathfrak{X}\rangle_\mathcal{A} = 0$ implies that $x = 0$, the same holding for $\langle \cdot, \cdot\rangle_\mathcal{A}$. Moreover, if the actions of $\mathcal{A}$ and $\mathcal{B}$ were strongly non-degenerate on $_\mathcal{B}\mathfrak{X}_\mathcal{A}$, this still holds for the quotient $\mathfrak{X}/N$. So, we can always find an equivalence bimodule $_\mathcal{B}\mathfrak{X}_\mathcal{A}$ with strongly non-degenerate $\mathcal{A}$ and $\mathcal{B}$ actions and inner products satisfying $\langle x, \mathfrak{X}\rangle_\mathcal{A} = 0 \implies x = 0$ and $_\mathcal{B}\langle x, \mathfrak{X}\rangle = 0 \implies x = 0$. We observe that, in this case (see proof of [9, Prop. 6.1]), $\mathcal{B} \cong \mathfrak{F}(\mathfrak{X}_\mathcal{A})$.

Given $_\mathcal{B}\mathfrak{X}_\mathcal{A}$ an equivalence bimodule with these properties, we note that if $x \cdot \mathcal{A} = 0$, then in particular $x \cdot \langle y, z\rangle_\mathcal{A} = 0$ for all $y, z \in \mathfrak{X}$. But then $_\mathcal{B}\langle x, y\rangle \cdot z = 0$ for all z, and hence $_\mathcal{B}\langle x, y\rangle = 0$. But since y is also arbitrary, it follows that $x = 0$. Therefore, $(\mathfrak{X}_\mathcal{A}, \langle \cdot, \cdot\rangle_\mathcal{A}) \in \mathsf{Mod}\text{-}\mathcal{A}$ and clearly establishes a Morita *-equivalence between $\mathcal{A}$ and $\mathfrak{F}(\mathfrak{X}_\mathcal{A}) \cong \mathcal{B}$. $\qquad\square$

We observe that *-algebras with approximate identities are non-degenerate and idempotent. We will now show the converse of the previous lemma in the case of C^*-algebras and Pedersen ideals (recalling that both types of algebras are also non-degenerate and idempotent). We start with some general lemmas.

Lemma 3.2 *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{P}_\mathcal{A}$ its Pedersen ideal. Then $A \in \mathcal{P}_\mathcal{A}$ is positive in $\mathcal{A}$ if and only if $A \in \mathcal{P}_\mathcal{A}{}^+$ (positivity in the purely algebraic sense of *-algebras over $\mathbb{C}$).*

PROOF: Let $A \in \mathcal{P}_\mathcal{A}$ be positive in $\mathcal{A}$ and let $\omega : \mathcal{P}_\mathcal{A} \longrightarrow \mathbb{C}$ be an arbitrary positive linear functional (in the algebraic sense, that is, $\omega(A^*A) \geq 0$ for all $A \in \mathcal{P}_\mathcal{A}$). We must show that $\omega(A) \geq 0$. Note that by the general properties of Pedersen ideals (see [25, Thm. 5.6.2]), if $A \in \mathcal{P}_\mathcal{A}$, then $C^*(A) \subseteq \mathcal{P}_\mathcal{A}$, where $C^*(A)$ denotes the C^*-algebra generated by A. Also observe that if $A \in \mathcal{A}^+$, then $A \in C^*(A)^+$, and hence $A = A_1^* A_1$, for some $A_1 \in C^*(A) \subseteq \mathcal{P}_\mathcal{A}$. Therefore, $\omega(A) = \omega(A_1^* A_1) \geq 0$. For the converse, note that if ω is a positive linear functional in $\mathcal{A}$, then $\omega|_{\mathcal{P}_\mathcal{A}}$ is positive in $\mathcal{P}_\mathcal{A}$ and hence if $A \in \mathcal{P}_\mathcal{A}^+$, we immediately have $\omega(A) \geq 0$. Hence $A \in \mathcal{A}^+$. $\qquad\square$

We now observe a general property of *-representations of Pedersen ideals on pre-Hilbert spaces which should be well-known.

Lemma 3.3 *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{P}_\mathcal{A}$ its Pedersen ideal and suppose $\pi : \mathcal{P}_\mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ is a *-representation on a complex pre-Hilbert space $\mathfrak{H}$. Then π extends to a *-representation $\pi^{\mathrm{cl}} : \mathcal{P}_\mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H}^{\mathrm{cl}})$, where $\mathfrak{H}^{\mathrm{cl}}$ is the closure of $\mathfrak{H}$. Moreover, $\|\pi^{\mathrm{cl}}(A)\| \leq \|A\|$ for all $A \in \mathcal{P}_\mathcal{A}$ and hence π^{cl} also extends to a *-representation of $\mathcal{A}$ on $\mathfrak{H}^{\mathrm{cl}}$.*

Remark 3.4 *As pointed out to us by P. Ara, the above lemma follows from a more general fact. If $\mathcal{A}$ is any complex *-algebra with positive definite involution ([15, pp. 338]), let $\mathcal{A}_b$ be the *-subalgebra of bounded elements of $\mathcal{A}$ ([15, pp. 338]). Then any *-representation of $\mathcal{A}_b$ on a complex pre-Hilbert space $\mathfrak{H}$ naturally extends to a *-representation of $\overline{\mathcal{A}_b}$ (closure of $\mathcal{A}_b$ with respect to its natural seminorm, as defined in [15, pp. 342]) on $\mathfrak{H}^{\mathrm{cl}}$.*

We can now discuss general positivity properties of modules over Pedersen ideals.

Proposition 3.5 *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{P}_{\mathcal{A}}$ its Pedersen ideal and $\mathcal{B}$ an arbitrary *-algebra over $\mathbb{C}$. Suppose $_{\mathcal{B}}\mathfrak{X}_{\mathcal{P}_{\mathcal{A}}}$ is a bimodule, equipped with a positive semi-definite $\mathcal{P}_{\mathcal{A}}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{P}_{\mathcal{A}}}$ and compatible $\mathcal{B}$-action. Then $\mathfrak{X}$ satisfies property $\boldsymbol{P}$.*

PROOF: We must show that given a complex pre-Hilbert space $\mathfrak{H}$ and a *-representation $\pi : \mathcal{P}_{\mathcal{A}} \longrightarrow \mathfrak{B}(\mathfrak{H})$, the formula $\langle x_1 \otimes \phi_1, x_2 \otimes \phi_2\rangle_{\widetilde{\mathfrak{K}}} = \langle \phi_1, \pi(\langle x_1, x_2\rangle_{\mathcal{P}_{\mathcal{A}}})\phi_2\rangle_{\mathfrak{H}}$, for $x_1, x_2 \in \mathfrak{X}$ and $\phi_1, \phi_2 \in \mathfrak{H}$ defines a positive semi-definite inner product on $\widetilde{\mathfrak{K}} = \mathfrak{X} \otimes_{\mathcal{P}_{\mathcal{A}}} \mathfrak{H}$. Note that since π extends to a *-representation $\pi^{\mathrm{cl}} : \mathcal{P}_{\mathcal{A}} \longrightarrow \mathfrak{B}(\mathfrak{H}^{\mathrm{cl}})$ on the Hilbert space completion of $\mathfrak{H}$, there is a natural isometric embedding $\iota : \mathfrak{X} \otimes_{\mathcal{P}_{\mathcal{A}}} \mathfrak{H} \longrightarrow \mathfrak{X} \otimes_{\mathcal{P}_{\mathcal{A}}} \mathfrak{H}^{\mathrm{cl}}$, where the $\mathcal{P}_{\mathcal{A}}$-balanced tensor products are defined using π and π^{cl}, respectively. So for $\phi_1, \dots, \phi_n \in \mathfrak{H} \subseteq \mathfrak{H}^{\mathrm{cl}}$ and $x_1, \dots, x_n \in \mathfrak{X}$, we have

$$\left\langle \sum_i x_i \otimes \phi_i, \sum_i x_i \otimes \phi_i \right\rangle_{\widetilde{\mathfrak{K}}} = \sum_{i,j} \left\langle \phi_i, \pi(\langle x_i, x_j\rangle_{\mathcal{A}})\phi_j \right\rangle_{\mathfrak{H}} = \sum_{i,j} \left\langle \phi_i, \pi^{\mathrm{cl}}(\langle x_i, x_j\rangle_{\mathcal{A}})\phi_j \right\rangle_{\mathfrak{H}^{\mathrm{cl}}}. \tag{3.1}$$

Therefore, to show the positivity of $\langle\cdot,\cdot\rangle_{\widetilde{\mathfrak{K}}}$, it suffices to show that the right hand side of (3.1) is non-negative. This can be done by the usual procedure (see [18, Ch. IV, Sect. 2.2]), using the fact that any *-representation of $\mathcal{P}_{\mathcal{A}}$ on a Hilbert space can be decomposed into the sum of a trivial representation and a non-degenerate one, and the non-degenerate part itself can be decomposed into the sum (in the topological sense) of cyclic (also in the topological sense) *-representations (see [18, Ch. I, Prop. 1.5.2]). $\qquad\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, let $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ be the corresponding Pedersen ideals and suppose there exists $\mathfrak{X} \in \mathsf{Mod}\text{-}\mathcal{P}_{\mathcal{A}}$, equipped with a pairing $\langle\cdot,\cdot\rangle$ so that $(\mathfrak{X}, \langle\cdot,\cdot\rangle)$ establishes a Morita *-equivalence between $\mathcal{P}_{\mathcal{A}}$ and $\mathfrak{F}(\mathfrak{X}_{\mathcal{P}_{\mathcal{A}}}) \cong \mathcal{P}_{\mathcal{B}}$. In [1, Thm. 2.4], it is shown that, in this case, one automatically has $\langle x, x\rangle \in \mathcal{A}^+$ and $\Theta_{x,x} \in \mathcal{B}^+$ (recall that for C^*-algebras, the usual notion of positivity coincides with positivity as defined for arbitrary *-algebras over C). Then, by Lemma 3.2, it follows that $\langle x, x\rangle \in \mathcal{P}_{\mathcal{A}}^+$ and $\Theta_{x,x} \in \mathcal{P}_{\mathcal{B}}^+$. It is then easy to check (see e.g. [9, Prop. 6.2]) that $\mathfrak{X}$ is a $(\mathcal{P}_{\mathcal{B}}\text{-}\mathcal{P}_{\mathcal{A}})$-bimodule satisfying all the properties of an equivalence bimodule, except possibly for $\boldsymbol{P}$ and $\boldsymbol{Q}$. But these properties follow directly from Proposition 3.5. Hence we have

Corollary 3.6 *If $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ are Morita *-equivalent, then they are automatically formally Morita equivalent.*

It then follows from Lemma 3.1, Proposition 3.5 and [1, Thm. 2.4] that

Theorem 3.7 *Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent if and only if their corresponding Pedersen ideals, $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$, are formally Morita equivalent. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are unital, then they are strongly Morita equivalent if and only if they are formally Morita equivalent.*

It follows from [9, Thm. 5.10] that if two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent, then $^*\text{-}\mathsf{Rep}(\mathcal{P}_{\mathcal{A}})$ and $^*\text{-}\mathsf{Rep}(\mathcal{P}_{\mathcal{B}})$ are equivalent categories. We recall from [28] that strong Morita equivalent C^*-algebras have equivalent categories of Hermitian modules. We conclude this section discussing how these two results are related.

For a C^*-algebra $\mathcal{A}$, let $\mathsf{hermod}(\mathcal{A})$ be the category of *-representations of $\mathcal{A}$ on Hilbert spaces, with isometric intertwiners as morphisms. The category of *non-degenerate* *-representations is denoted by $\mathsf{Hermod}(\mathcal{A})$ and objects in this category are called *Hermitian modules* [27]. In order to understand how these categories are related to the categories of purely algebraic representations *-$\mathsf{rep}(\mathcal{P}_{\mathcal{A}})$ and *-$\mathsf{Rep}(\mathcal{P}_{\mathcal{A}})$, we introduce two other categories. By *-$\mathsf{repDS}(\mathcal{A})$ we denote the category of *-representations of $\mathcal{A}$ on Hilbert spaces with a dense $\mathcal{P}_{\mathcal{A}}$-invariant subspace, while for *-$\mathsf{RepDS}(\mathcal{A})$ we require in addition that the induced *-representation of $\mathcal{P}_{\mathcal{A}}$ restricted to the dense subspace be strongly non-degenerate (in the algebraic sense). For the morphisms we take isometric intertwiners preserving the dense invariant subspaces. Note that objects in *-$\mathsf{RepDS}(\mathcal{A})$ are in particular Hermitian modules over $\mathcal{A}$ (with a distinguished dense subspace). One can check that the process of extending the *-representations, discussed in Lemma 3.3, gives rise to a functor from *-$\mathsf{Rep}(\mathcal{P}_{\mathcal{A}})$ to *-$\mathsf{RepDS}(\mathcal{A})$ (*-$\mathsf{rep}(\mathcal{P}_{\mathcal{A}})$ to *-$\mathsf{repDS}(\mathcal{A})$) for which the natural restriction functor is an inverse. It then follows that *-$\mathsf{Rep}(\mathcal{P}_{\mathcal{A}})$ and *-$\mathsf{RepDS}(\mathcal{A})$ (*-$\mathsf{rep}(\mathcal{P}_{\mathcal{A}})$ and *-$\mathsf{repDS}(\mathcal{A})$) are equivalent categories. We point out that *-$\mathsf{RepDS}(\mathcal{A})$ and $\mathsf{Hermod}(\mathcal{A})$ (*-$\mathsf{repDS}(\mathcal{A})$ and $\mathsf{hermod}(\mathcal{A})$) are not equivalent categories in general. We finally remark that the categories *-$\mathsf{repDS}(\mathcal{A})$ and *-$\mathsf{RepDS}(\mathcal{A})$ are interesting in their own right since, in many applications, *-representations of $\mathcal{A}$ come automatically with a dense invariant subspace, as in GNS representations. As another example, certain unbounded operators associated to *-representations, such as generators of symmetries, distinguish dense subspaces by their domains of definition.

4   Closed *-Ideals and the Minimal Ideal

After having clarified the connection between formal Morita equivalence and C^*-algebras, we consider in this section general *-algebras over C and start the discussion about (closed) *-ideals.

If $\mathcal{I} \subseteq \mathcal{A}$ is a *-ideal, i.e. an ideal closed under the *-involution, then $\mathcal{A}/\mathcal{I}$ is again a *-algebra. If $\mathcal{I}$ is a left (or right) ideal which is closed under the *-involution then it is automatically a two-sided ideal and hence a *-ideal.

Definition 4.1 *A *-ideal $\mathcal{I}$ of $\mathcal{A}$ is called closed if it is the kernel of a *-representation.*

Clearly, the kernel $\ker \pi$ of a *-representation π is a *-ideal, and unitary equivalent *-representations have the same kernel. But $\ker \pi$ does not determine this equivalence class since e.g. $\ker \pi = \ker(\pi \oplus \pi)$. We observe that if $\mathcal{A}$ is a C^*-algebra, then a *-ideal $\mathcal{I}$ in $\mathcal{A}$ is closed in this algebraic sense if and only if it is norm closed. Indeed, recall that a *-ideal in a C^*-algebra is norm closed if and only if it is the kernel of some *-representation of $\mathcal{A}$ on a Hilbert space ([18, Ch. I, Thm. 1.3.10]). But since any representation of a C^*-algebra on a pre-Hilbert space extends to a representation on its completion (see Lemma 3.3), the conclusion follows.

Lemma 4.2 *Let $\mathcal{A}$ be a *-algebra over C and $\mathcal{I}$ a *-ideal. Then $\mathcal{A}/\mathcal{I}$ has a faithful *-representation if and only if $\mathcal{I}$ is closed. If $\{\mathcal{I}_{\alpha}\}_{\alpha \in \Lambda}$ are closed *-ideals then $\mathcal{I} = \bigcap_{\alpha \in \Lambda} \mathcal{I}_{\alpha}$ is again a closed *-ideal.*

PROOF: For the first part use the projection $\rho : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ to pull-back *-representations. For the second observe that if $\mathcal{I}_{\alpha} = \ker \pi_{\alpha}$ then $\mathcal{I} = \ker(\bigoplus_{\alpha \in \Lambda} \pi_{\alpha})$. $\qquad\square$

Since formal Morita equivalence deals with strongly non-degenerate *-representations, it would be nice if these particular representations were sufficient to define all closed *-ideals. If $\mathcal{A}$ is idempotent, then this is indeed the case.

Lemma 4.3 *Let $\mathcal{A}$ be an idempotent *-algebra over C and $\pi : \mathcal{A} \to \mathfrak{B}(\mathfrak{H})$ a *-representation. Let*

$$\mathfrak{H}_{\mathrm{snd}} := \big\{ \phi \in \mathfrak{H} \mid \exists \phi_i \in \mathfrak{H}, A_i \in \mathcal{A}, i = 1, \dots, n : \phi = \textstyle\sum_i \pi(A_i)\phi_i \big\}. \qquad (4.1)$$

Then $\mathfrak{H}_{\mathrm{snd}}$ is π-invariant and the restriction $\pi_{\mathrm{snd}} := \pi|_{\mathfrak{H}_{\mathrm{snd}}}$ is strongly non-degenerate with $\ker \pi_{\mathrm{snd}} = \ker \pi$. Moreover, every GNS representation is strongly non-degenerate.

PROOF: The invariance of $\mathfrak{H}_{\mathrm{snd}}$ is obvious and the fact that π_{snd} is strongly non-degenerate follows from the idempotency of $\mathcal{A}$. Finally let $\pi_{\mathrm{snd}}(A) = 0$. Then in particular $\pi(A)\pi(A^*)\phi = 0$ for all $\phi \in \mathfrak{H}$ whence $\pi(A) = 0$ follows and thus $\ker \pi_{\mathrm{snd}} \subseteq \ker \pi$. The other inclusion is trivial. The strong non-degeneracy of a GNS representation follows directly from the definition. $\qquad\square$

Using the closed *-ideals, we define the following closure operation. If $\mathcal{J} \subseteq \mathcal{A}$ is an arbitrary subset then $\mathcal{J}^{\mathrm{cl}}$ is the smallest closed *-ideal containing $\mathcal{J}$, i.e.

$$\mathcal{J}^{\mathrm{cl}} := \bigcap_{\mathcal{J}\subseteq\mathcal{I},\, \mathcal{I}\ \mathrm{closed}\ ^*\text{-ideal}} \mathcal{I}, \tag{4.2}$$

where $\mathcal{I}$ runs over the closed *-ideals of $\mathcal{A}$. Clearly $\mathcal{J} \subseteq \mathcal{J}^{\mathrm{cl}}$ and $(\mathcal{J}^{\mathrm{cl}})^{\mathrm{cl}} = \mathcal{J}^{\mathrm{cl}}$ for any subset of $\mathcal{A}$. This implies that for $\mathcal{J} \subseteq \mathcal{I}$ we have $\mathcal{J}^{\mathrm{cl}} \subseteq \mathcal{I}^{\mathrm{cl}}$ and $(\bigcup_{\alpha\in\Lambda} \mathcal{I}_\alpha)^{\mathrm{cl}} = (\bigcup_{\alpha\in\Lambda} \mathcal{I}_\alpha^{\mathrm{cl}})^{\mathrm{cl}}$ for an arbitrary index set Λ and $\mathcal{I}_\alpha \subseteq \mathcal{A}$. Finally note that $\bigcap_{\alpha\in\Lambda} \mathcal{I}_\alpha^{\mathrm{cl}} = (\bigcap_{\alpha\in\Lambda} \mathcal{I}_\alpha^{\mathrm{cl}})^{\mathrm{cl}}$. We can define the following operations. For two subsets $\mathcal{I}, \mathcal{J} \subseteq \mathcal{A}$, we set

$$\mathcal{I} \vee \mathcal{J} := (\mathcal{I} \cup \mathcal{J})^{\mathrm{cl}} \quad \text{and} \quad \mathcal{I} \wedge \mathcal{J} := \mathcal{I}^{\mathrm{cl}} \cap \mathcal{J}^{\mathrm{cl}}. \tag{4.3}$$

One can check that these two operations define a lattice structure on the set of closed *-ideals of $\mathcal{A}$. In this case, the operation '$\leq$', which is defined by $\mathcal{I} \leq \mathcal{J}$ if $\mathcal{I} \wedge \mathcal{J} = \mathcal{I}$, coincides with the set-theoretic '$\subseteq$'. We then have a lattice structure on the set of all closed *-ideals in $\mathcal{A}$, that we denote by $\mathfrak{L}(\mathcal{A})$. Note that all *-ideals, closed or not, also have a lattice structure, with the 'closure map' defined by taking the smallest *-ideal containing the corresponding subset. However, we will be mainly interested in the closed *-ideals.

For a C^*-algebra $\mathcal{A}$, $\mathfrak{L}(\mathcal{A})$ coincides with its usual lattice of norm closed *-ideals. If $\mathcal{P}_\mathcal{A}$ is its Pedersen ideal, note that we have a natural map $F : \mathfrak{L}(\mathcal{P}_\mathcal{A}) \to \mathfrak{L}(\mathcal{A})$ defined as follows. Let $\mathcal{I} \in \mathfrak{L}(\mathcal{P}_\mathcal{A})$. Then $\mathcal{I} = \ker \pi$, for some *-representation of $\mathcal{P}_\mathcal{A}$ on a pre-Hilbert space $\mathfrak{H}$. As we observed in Lemma 3.3, π extends to a *-representation π^{cl} of $\mathcal{A}$ on $\mathfrak{H}^{\mathrm{cl}}$. We then define F as the map $\mathcal{I} = \ker \pi \mapsto F(\mathcal{I}) = \ker \pi^{\mathrm{cl}}$.

Proposition 4.4 $F : \mathfrak{L}(\mathcal{P}_\mathcal{A}) \longrightarrow \mathfrak{L}(\mathcal{A})$ *is a lattice isomorphism.*

PROOF: Let us first observe that F is a well-defined map. Note that if π is a *-representation of $\mathcal{P}_\mathcal{A}$, then $\ker \pi$ is always norm dense in $\ker \pi^{\mathrm{cl}}$. To see that, let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate identity in $\mathcal{A}$, with $e_\lambda \in \mathcal{P}_\mathcal{A}$. This is possible since $\mathcal{P}_\mathcal{A}$ is dense in $\mathcal{A}$ ([12, Thm. I.7.2]). Then given $A \in \ker \pi^{\mathrm{cl}}$, we have $e_\lambda A \in \ker \pi^{\mathrm{cl}} \bigcap \mathcal{P}_\mathcal{A} = \ker \pi$ and $e_\lambda A \to A$. Now let $\mathcal{I} \in \mathfrak{L}(\mathcal{P}_\mathcal{A})$ and suppose π and ρ are two different *-representations of $\mathcal{P}_\mathcal{A}$ so that $\mathcal{I} = \ker \pi = \ker \rho$. We must show that $\ker \pi^{\mathrm{cl}} = \ker \rho^{\mathrm{cl}}$. Note that if $A \in \ker \pi^{\mathrm{cl}}$, then we can find $A_\lambda \in \ker \pi = \ker \rho$ with $A_\lambda \to A$. So $\rho(A_\lambda) = 0$, for all λ, and hence $\rho(A) = 0$. This shows that $\ker \pi^{\mathrm{cl}} \subseteq \ker \rho^{\mathrm{cl}}$ and by changing the roles of π and ρ we conclude that $\ker \pi^{\mathrm{cl}} = \ker \rho^{\mathrm{cl}}$. This proves well-definedness. It is not hard to check that F is a bijection which preserves the order structure of the lattices. This guarantees that F is a lattice isomorphism. $\qquad\square$

Recall that for a C^*-algebra, any (topologically) non-degenerate *-representation can be decomposed into a direct sum (also in the topological sense) of GNS representations. Thus for C^*-algebras, any closed *-ideal can be obtained as the kernel of a (topological) sum of GNS representations. We observe that for a general *-algebra over C, it is not true that an arbitrary (strongly non-degenerate) *-representation decomposes into an algebraic sum of GNS representations. In fact, even if C is

an algebraically closed field (which is complete in its canonical topology) and we allow topological sums, this decomposition does not hold in general. This follows since otherwise every Hilbert space over C would have a Hilbert basis (this notion makes sense in this situation), which is *not* the case (see e.g. [8, Thm. 7]). Nevertheless, it is still true that arbitrary closed *-ideals on a *-algebra arise as kernel of sums of GNS representations, as we now observe.

Let $\omega : \mathcal{A} \to \mathsf{C}$ be a positive linear functional. For $B \in \mathcal{A}$, we consider the positive linear functional $\omega_B(A) := \omega(B^*AB) = \langle \psi_B, \pi_\omega(A)\psi_B \rangle$. Then $A \in \ker \pi_\omega$ if and only if $\omega_B(A) = 0$ for all $B \in \mathcal{A}$. Using a polarization argument and the fact that $\mathfrak{B}(\mathfrak{H}_\omega)$ is torsion-free ([9, Prop. 2.8]), we see that this is equivalent to $\omega_B(A^*A) = 0$ for all $B \in \mathcal{A}$. Hence we have

$$\ker \pi_\omega = \bigcap_{B \in \mathcal{A}} \mathfrak{I}_{\omega_B} = \bigcap_{B \in \mathcal{A}} \ker \omega_B. \tag{4.4}$$

Note that neither $\mathfrak{I}_{\omega_B}$ nor $\ker \omega_B$ is a *-ideal in general but only the above intersection. Now consider an arbitrary *-representation π on $\mathfrak{H}$. For every $\varphi \in \mathfrak{H}$, one defines the expectation value $\omega_\varphi(A) := \langle \varphi, \pi(A)\varphi \rangle$, which is clearly a positive linear functional of $\mathcal{A}$. Denoting the Gel'fand ideal by $\mathfrak{I}_\varphi$ and the corresponding GNS representation by π_φ, it follows that $A \in \ker \pi_\varphi$ if and only if $\pi(A)\pi(B)\varphi = 0$ for all $B \in \mathcal{A}$. But this implies that $A \in \mathfrak{I}_\varphi$ by taking $B = A^*$. Moreover, $A \in \mathfrak{I}_\varphi$ if and only if $\pi(A)\varphi = 0$. So $A \in \ker \omega_\varphi$. Thus we have the inclusions

$$\ker \pi_\varphi \subseteq \mathfrak{I}_\varphi \subseteq \ker \omega_\varphi. \tag{4.5}$$

Lemma 4.5 *Let $\mathcal{A}$ be a *-algebra over C and $\pi : \mathcal{A} \to \mathfrak{B}(\mathfrak{H})$ a *-representation. Consider $\pi_M := \bigoplus_{\varphi \in \mathfrak{H}} \pi_\varphi$. Then one has*

$$\ker \pi = \ker \pi_M = \bigcap_{\varphi \in \mathfrak{H}} \ker \pi_\varphi = \bigcap_{\varphi \in \mathfrak{H}} \mathfrak{I}_\varphi = \bigcap_{\varphi \in \mathfrak{H}} \ker \omega_\varphi. \tag{4.6}$$

PROOF: It remains to show $\bigcap_{\varphi \in \mathfrak{H}} \ker \omega_\varphi = \ker \pi$. But this follows by polarization and the fact that $\mathfrak{B}(\mathfrak{H})$ is torsion-free. $\qquad\qquad\square$

Lemma 4.6 *Let $\mathcal{A}$ be a *-algebra over C with an approximate identity. Then for every positive linear functional ω one has*

$$\ker \pi_\omega \subseteq \mathfrak{I}_\omega \subseteq \ker \omega. \tag{4.7}$$

PROOF: This generalizes (4.5). Let $A \in \mathcal{A}$ and $E_\alpha = E_\alpha^* \in \mathcal{A}$ such that $E_\alpha A = A = AE_\alpha$. Then $\pi_\omega(A) = 0$ implies $0 = \pi_\omega(A)\psi_{E_\alpha} = \psi_A$, whence $A \in \mathfrak{I}_\omega$. For $A \in \mathfrak{I}_\omega$, we find by the Cauchy-Schwarz inequality that $0 \leq \omega(A)\overline{\omega(A)} \leq \omega(A^*A)\omega(E_\alpha^*E_\alpha) = 0$ and hence $A \in \ker \omega$. $\qquad\qquad\square$

Let us investigate the lattice of closed *-ideals more closely. While the lattice of *-ideals trivially has a minimal and a maximal element, namely $\{0\}$ and $\mathcal{A}$, the closed *-ideals provide a more interesting structure. We define the minimal closed *-ideal of $\mathcal{A}$ by

$$\mathfrak{I}_{\min}(\mathcal{A}) := \bigcap_{\mathfrak{I} \text{ closed *-ideal}} \mathfrak{I}, \tag{4.8}$$

which is again a closed *-ideal due to Lemma 4.2. Furthermore $\mathfrak{I}_{\min}(\mathcal{A})$ is clearly the minimal element of the lattice. It is obvious that $\mathfrak{I}_{\min}(\mathcal{A}) = \{0\}$ if and only if $\mathcal{A}$ has a faithful *-representation

on some pre-Hilbert space. Thus for a C^*-algebra $\mathcal{A}$ one automatically has $\mathcal{J}_{\min}(\mathcal{A}) = \{0\}$. We note that the same holds for star-product algebras (see e.g. [32, Sect. 4]).

Since the kernel of a *-representation can always be obtained by using direct sums of GNS representations, it is sufficient to consider GNS representations in order to characterize $\mathcal{J}_{\min}(\mathcal{A})$. With (4.4), this gives

$$\mathcal{J}_{\min}(\mathcal{A}) = \bigcap_\omega \ker \pi_\omega = \bigcap_\omega \bigcap_{B \in \mathcal{A}} \mathcal{J}_{\omega_B} = \bigcap_\omega \bigcap_{B \in \mathcal{A}} \ker \omega_B, \tag{4.9}$$

where ω runs over all positive linear functionals. If $\mathcal{A}$ has in addition an approximate identity, we obtain the following result:

Theorem 4.7 *Let $\mathcal{A}$ be a *-algebra over C with approximate identity. Then the minimal closed *-ideal is given by*

$$\mathcal{J}_{\min}(\mathcal{A}) = \bigcap_\omega \ker \pi_\omega = \bigcap_\omega \mathcal{J}_\omega = \bigcap_\omega \ker \omega, \tag{4.10}$$

where ω runs over all positive linear functionals. If $A^*A = 0$, or if A is normal and nilpotent, or if $zA = 0$ for $0 \neq z \in \mathsf{C}$ then $A \in \mathcal{J}_{\min}(\mathcal{A})$.

PROOF: Let $A \in \bigcap_\omega \ker \omega$. Then $\pi(A) = 0$ for all *-representations π since otherwise there would exist a $\varphi \in \mathfrak{H}$ such that $\omega_\varphi(A) = \langle \varphi, \pi(A)\varphi \rangle \neq 0$ which contradicts the assumption since ω_φ is a positive functional. Hence $A \in \mathcal{J}_{\min}(\mathcal{A})$ and by (4.7) and (4.9) the equality (4.10) follows. The other statements follow directly from [9, Prop. 2.8]. $\square$

Note that all 'non-C^*-algebra-like' elements are absorbed into $\mathcal{J}_{\min}(\mathcal{A})$. In particular $\mathcal{A}/\mathcal{J}_{\min}(\mathcal{A})$ has sufficiently many positive linear functionals and still an approximate identity. This can be seen as an analogue of the construction of the C^*-enveloping algebra of a Banach *-algebra, see e.g. [12, Sect. II.7]. The passage from $\mathcal{A}$ to $\mathcal{A}/\mathcal{J}_{\min}(\mathcal{A})$ is functorial in the category of *-algebras with approximate identities.

Proposition 4.8 *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over C with approximate identities and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a *-homomorphism. Then $\Phi(\mathcal{J}_{\min}(\mathcal{A})) \subseteq \mathcal{J}_{\min}(\mathcal{B})$ and thus there exists a unique *-homomorphism $\phi : \mathcal{A}/\mathcal{J}_{\min}(\mathcal{A}) \to \mathcal{B}/\mathcal{J}_{\min}(\mathcal{B})$ such that*

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Phi} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{A}/\mathcal{J}_{\min}(\mathcal{A}) & \xrightarrow{\phi} & \mathcal{B}/\mathcal{J}_{\min}(\mathcal{B})
\end{array}
\tag{4.11}
$$

commutes. Thus the passage from $\mathcal{A}$ to $\mathcal{A}/\mathcal{J}_{\min}(\mathcal{A})$ is functorial.

PROOF: If $\omega : \mathcal{B} \to \mathsf{C}$ is a positive linear functional then $\Phi^*\omega = \omega \circ \Phi : \mathcal{A} \to \mathsf{C}$ is positive, too, whence $\Phi(\mathcal{J}_{\min}(\mathcal{A})) \subseteq \mathcal{J}_{\min}(\mathcal{B})$ follows from (4.10). Then (4.11) is clear. $\square$

Remark 4.9 *The minimal ideal as discussed here is related to the work of Handelman [15], as P. Ara brought to our attention. Let $\mathcal{A}$ be a (unital) *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with positive definite involution ([15, pp. 338]) and suppose $\mathbb{Q} \subseteq \mathsf{R} \subseteq \mathbb{R}$ (so that R is archimedean-ordered). Let $J^*(\mathcal{A})$ be the set of bounded elements of $\mathcal{A}$ with zero seminorm ([15, Sect.1]). We note that, in general, $J^*(\mathcal{A}) \subseteq \mathcal{J}_{\min}(\mathcal{A}_b)$. But if $\mathsf{C} = \mathbb{C}$, then in fact $J^*(\mathcal{A}) = \mathcal{J}_{\min}(\mathcal{A}_b)$, since in this case the *-algebra $\mathcal{A}_b/J^*(\mathcal{A})$ can be completed to a C^*-algebra and hence admits a faithful *-representation.*

5 Morita Equivalent *-Algebras

In this section, we shall discuss the behavior of the lattice of (closed) *-ideals under algebraic Rieffel induction and prove that the lattice of (closed) *-ideals is a 'formal Morita invariant'.

Let $_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule with an $\mathcal{A}$-valued inner product (see Section 2). We define the map $\Phi_{\mathfrak{X}} : 2^{\mathcal{A}} \to 2^{\mathcal{B}}$ by

$$\Phi_{\mathfrak{X}}(\mathfrak{I}) := \{B \in \mathcal{B} \mid \forall x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}} : \langle x, B \cdot y\rangle_{\mathcal{A}} \in \mathfrak{I}\}. \tag{5.1}$$

If we consider (closed) *-ideals of $\mathcal{A}$, then we obtain the following properties of $\Phi_{\mathfrak{X}}$:

Proposition 5.1 *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over $\mathbb{C}$ and let $_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule with $\mathcal{A}$-valued inner product. Then $\Phi_{\mathfrak{X}}$ maps *-ideals of $\mathcal{A}$ to *-ideals of $\mathcal{B}$ and satisfies*

$$\Phi_{\mathfrak{X}}(\mathfrak{I}) \wedge \Phi_{\mathfrak{X}}(\mathfrak{J}) = \Phi_{\mathfrak{X}}(\mathfrak{I} \wedge \mathfrak{J}) \qquad and \qquad \mathfrak{I} \leq \mathfrak{J} \Longrightarrow \Phi_{\mathfrak{X}}(\mathfrak{I}) \leq \Phi_{\mathfrak{X}}(\mathfrak{J}) \tag{5.2}$$

*for all *-ideals $\mathfrak{I}$, $\mathfrak{J}$ of $\mathcal{A}$. If in addition the inner product is positive semi-definite and satisfies $\boldsymbol{P}$, then $\Phi_{\mathfrak{X}}$ maps closed *-ideals to closed *-ideals, satisfies (5.2) with respect to the lattice of closed *-ideals, and one has*

$$\ker(\mathfrak{R}_{\mathfrak{X}}\pi_{\mathcal{A}}) = \Phi_{\mathfrak{X}}(\ker \pi_{\mathcal{A}}) \tag{5.3}$$

*for any *-representation $\pi_{\mathcal{A}}$ of $\mathcal{A}$.*

PROOF: The first part is a simple computation thus assume that the inner product is positive semi-definite and satisfies $\boldsymbol{P}$. Let $\pi_{\mathcal{A}}$ be a *-representation of $\mathcal{A}$ on $\mathfrak{H}$ and let $\pi_{\mathcal{B}} = \mathfrak{R}_{\mathfrak{X}}\pi_{\mathcal{A}}$ be the induced *-representation of $\mathcal{B}$. Then $\pi_{\mathcal{B}}(B) = 0$ if and only if $[B \cdot y \otimes \phi] = 0$ for all $y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and $\phi \in \mathfrak{H}$ since the equivalence classes of those $y \otimes \phi$ span $\mathfrak{K}$. But since the inner product of $\mathfrak{K}$ is non-degenerate this is equivalent to $0 = \langle[x \otimes \psi], [B \cdot y \otimes \phi]\rangle_{\mathfrak{K}} = \langle\psi, \pi_{\mathcal{A}}(\langle x, B \cdot y\rangle_{\mathcal{A}})\phi\rangle_{\mathfrak{H}}$ for all $x, y \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ and $\phi, \psi \in \mathfrak{H}$. Thus it is equivalent to $\pi_{\mathcal{A}}(\langle x, B \cdot y\rangle_{\mathcal{A}}) = 0$ and hence $B \in \Phi_{\mathfrak{X}}(\ker \pi_{\mathcal{A}})$ which implies (5.3). Hence closed *-ideals are mapped to closed *-ideals and since $\wedge$ and $\leq$ are given by $\cap$ and $\subseteq$ the proof is completed. $\square$

Note that from (5.2) one obtains that $\Phi_{\mathfrak{X}}$ is almost a homomorphism of lattices, i.e. we have

$$\Phi_{\mathfrak{X}}(\mathfrak{I}) \vee \Phi_{\mathfrak{X}}(\mathfrak{J}) \leq \Phi_{\mathfrak{X}}(\mathfrak{I} \vee \mathfrak{J}) \tag{5.4}$$

for all *-ideals or closed *-ideals, respectively. Note that in general we only can guarantee '$\leq$' since there may exist more and perhaps smaller (closed) *-ideals of $\mathcal{B}$ which are not in the image of $\Phi_{\mathfrak{X}}$.

In order to discuss Morita equivalence, we need the following technical lemma, which does not use any positivity requirements but only the *-structures.

Lemma 5.2 *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over $\mathbb{C}$ and $_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ a $(\mathcal{B}\text{-}\mathcal{A})$-bimodule with compatible $\mathcal{A}$- and $\mathcal{B}$-valued inner products and let $\mathfrak{I} \subseteq \mathcal{A}$ be a *-ideal. Then $A \in \Phi_{\overline{\mathfrak{X}}}(\Phi_{\mathfrak{X}}(\mathfrak{I}))$ if and only if for all $x, y, z, w \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ one has*

$$\langle x, y\rangle_{\mathcal{A}} A \langle z, w\rangle_{\mathcal{A}} \in \mathfrak{I}. \tag{5.5}$$

PROOF: By definition $A \in \Phi_{\overline{\mathfrak{X}}}(\Phi_{\mathfrak{X}}(\mathfrak{I}))$ if and only if one has $\langle y, \langle \overline{x}, A \cdot \overline{z}\rangle_{\mathcal{B}} \cdot w\rangle_{\mathcal{A}} \in \mathfrak{I}$ for all $\overline{x}, \overline{z} \in {}_{\mathcal{A}}\overline{\mathfrak{X}}_{\mathcal{B}}$ and $y, w \in {}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$. Then it is a simple computation to obtain the condition (5.5). $\square$

If in addition the $\mathcal{A}$-valued inner product is full, then condition (5.5) means that the ideal generated by A must be contained in $\mathfrak{I}$. If furthermore $\mathcal{A}$ has an approximate identity, then we have $\Phi_{\overline{\mathfrak{X}}}(\Phi_{\mathfrak{X}}(\mathfrak{I})) = \mathfrak{I}$ for all *-ideals $\mathfrak{I}$. By symmetry in $\mathcal{A}$ and $\mathcal{B}$, we end up with the following proposition.

Proposition 5.3 *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over C with approximate identities and let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be a $(\mathcal{B}$-$\mathcal{A})$-bimodule with compatible full $\mathcal{B}$- and $\mathcal{A}$-valued inner products. Then the lattices of *-ideals are isomorphic via $\Phi_{\mathfrak{X}}$ and $\Phi_{\overline{\mathfrak{X}}}$.*

PROOF: The fact that $\Phi_{\mathfrak{X}}$ and $\Phi_{\overline{\mathfrak{X}}}$ are inverse to each other follows from the above considerations. Hence it remains to show that $\Phi_{\mathfrak{X}}$ (and thus $\Phi_{\overline{\mathfrak{X}}}$) is a lattice homomorphism. But this follows from bijectivity and Proposition 5.1. $\square$

In order to make statements about closed *-ideals, we need the positivity requirements for the inner products to guarantee that $\mathfrak{R}_{\mathfrak{X}}\pi_{\mathcal{A}}$ is indeed a *-representation.

Theorem 5.4 *Let $\mathcal{A}$, $\mathcal{B}$ be idempotent *-algebras over C and let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be an equivalence bimodule. Then the lattices of closed *-ideals are isomorphic via $\Phi_{\mathfrak{X}}$ and $\Phi_{\overline{\mathfrak{X}}}$.*

PROOF: Let $\mathcal{I} = \ker\pi_{\mathcal{A}}$. We may assume, due to Lemma 4.3, that $\pi_{\mathcal{A}}$ is strongly non-degenerate. Hence, by [9, Lem. 5.7], we know that $\mathfrak{R}_{\overline{\mathfrak{X}}}\mathfrak{R}_{\mathfrak{X}}\pi_{\mathcal{A}}$ is unitarily equivalent to $\pi_{\mathcal{A}}$. So they have the same kernel and by (5.3) it follows that $\Phi_{\overline{\mathfrak{X}}}(\Phi_{\mathfrak{X}}(\mathcal{I})) = \mathcal{I}$. Analogously, one obtains $\Phi_{\mathfrak{X}}(\Phi_{\overline{\mathfrak{X}}}(\mathcal{J})) = \mathcal{J}$ for all closed *-ideals of $\mathcal{B}$ and hence $\Phi_{\mathfrak{X}}$ and $\Phi_{\overline{\mathfrak{X}}}$ are inverse to each other. The homomorphism properties follow as in Proposition 5.3. $\square$

Note that the so-called Rieffel correspondence theorem ([26, Thm. 3.24]) can be recovered as a consequence of the above theorem. Indeed, if $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent C^*-algebras, then by Theorem 3.7 it follows that $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ are formally Morita equivalent. So $\mathfrak{L}(\mathcal{P}_{\mathcal{A}})$ and $\mathfrak{L}(\mathcal{P}_{\mathcal{B}})$ are isomorphic by Theorem 5.4 and therefore $\mathfrak{L}(\mathcal{A})$ and $\mathfrak{L}(\mathcal{B})$ are isomorphic by Proposition 4.4. We also observe the following corollary.

Corollary 5.5 *Let $\mathcal{A}$ and $\mathcal{B}$ be non-degenerate and idempotent *-algebras over C which are formally Morita equivalent. Then $\mathcal{J}_{\min}(\mathcal{A}) = 0$ if and only if $\mathcal{J}_{\min}(\mathcal{B}) = 0$.*

PROOF: Let ${}_{\mathcal{B}}\mathfrak{X}_{\mathcal{A}}$ be an equivalence bimodule. Note that if $\mathcal{A}$ is non-degenerate and idempotent, then ${}_{\mathcal{B}}\langle x, y \cdot A\rangle = 0$ for all $x, y \in \mathfrak{X}$ implies that $A = 0$ (see the proof of Lemma 3.1). Hence the conclusion follows. $\square$

Let us finally discuss the general phenomena of *-algebras having equivalent (even isomorphic) categories of (strongly non-degenerate) *-representations but which are not formally Morita equivalent. The following discussion generalizes the example of C and $\bigwedge^{\bullet}(\mathsf{C}^n)$ from [9, Cor. 5.20].

Consider the categories of *-representations of $\mathcal{A}$ and $\mathcal{A}/\mathcal{J}_{\min}(\mathcal{A})$. More generally, we shall consider a *-ideal $\mathcal{J} \subseteq \mathcal{J}_{\min}(\mathcal{A})$ and the *-algebra $\mathcal{A}/\mathcal{J}$. Denoting by $\rho : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ the *-homomorphism $A \mapsto [A]$ we can pull-back *-representations $\pi : \mathcal{A}/\mathcal{J} \to \mathfrak{B}(\mathfrak{H})$ by

$$\rho^*\pi(A) := \pi(\rho(A)) = \pi([A]), \tag{5.6}$$

and obtain a *-representation $\rho^*\pi$ of $\mathcal{A}$ on the same pre-Hilbert space $\mathfrak{H}$. This is clearly functorial, i.e. compatible with intertwiners. Moreover, if π is also strongly non-degenerate then $\rho^*\pi$ is strongly non-degenerate. On the other hand, let $\pi : \mathcal{A} \to \mathfrak{B}(\mathfrak{H})$ be a *-representation of $\mathcal{A}$ then $\mathcal{J} \subseteq \mathcal{J}_{\min}(\mathcal{A}) \subseteq \ker\pi$. Hence we can also push-forward the *-representation and obtain a *-representation $\rho_*\pi$ of $\mathcal{A}/\mathcal{J}$ on the same representation space

$$\rho_*\pi([A]) := \pi(A). \tag{5.7}$$

Again this is functorial and if π is strongly non-degenerate then $\rho_*\pi$ is also strongly non-degenerate. Collecting these results, we obtain

Proposition 5.6 *Let $\mathcal{A}$ be a *-algebra over C and $\mathcal{J} \subseteq \mathcal{J}_{\min}(\mathcal{A})$ a *-ideal. Then the functors ρ^* and ρ_* implement an isomorphism of the categories *-$\mathsf{rep}(\mathcal{A})$ and *-$\mathsf{rep}(\mathcal{A}/\mathcal{J})$ as well as of *-$\mathsf{Rep}(\mathcal{A})$ and *-$\mathsf{Rep}(\mathcal{A}/\mathcal{J})$.*

We also note that $\mathcal{A}$ and $\mathcal{A}/\mathcal{J}$ have isomorphic lattices of closed *-ideals, for any *-ideal $\mathcal{J} \subseteq \mathcal{J}_{\min}(\mathcal{A})$. Let $\mathcal{I} = \ker \pi \in \mathfrak{L}(\mathcal{A})$. Since $\mathcal{J} \subseteq \mathcal{J}_{\min}(\mathcal{A}) \subseteq \ker \pi$, it follows that $\rho(\mathcal{I}) = \ker \rho_* \pi$. Thus ρ induces a map $\mathfrak{L}(\mathcal{A}) \to \mathfrak{L}(\mathcal{A}/\mathcal{J})$ which can be easily checked to be a bijection which preserves ordering, and hence is a lattice isomorphism. We can then state

Proposition 5.7 *Let $\mathcal{A}$ be a *-algebra over C and $\mathcal{J} \subseteq \mathcal{J}_{\min}(\mathcal{A})$ a *-ideal. Then the natural projection $\rho : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ induces an isomorphism of $\mathfrak{L}(\mathcal{A})$ and $\mathfrak{L}(\mathcal{A}/\mathcal{J})$. In particular, $\mathcal{J}_{\min}(\mathcal{A}/\mathcal{J}) = \mathcal{J}_{\min}(\mathcal{A})/\mathcal{J}$.*

We finally observe that this discussion provides a large class of examples of *-algebras over C having equivalent categories of *-representations, isomorphic lattice of *-ideals but which are not formally Morita equivalent.

Proposition 5.8 *Let $\mathcal{A}$ be a *-algebra over C with approximate identity and let $\mathcal{I}$ be a *-ideal contained in $\mathcal{J}_{\min}(\mathcal{A})$ but $\mathcal{I} \neq \mathcal{J}_{\min}(\mathcal{A})$. Then $\mathcal{A}/\mathcal{I}$ and $\mathcal{A}/\mathcal{J}_{\min}(\mathcal{A})$ have isomorphic categories of (strongly non-degenerate) *-representations and isomorphic lattices of closed *-ideals, but they are not formally Morita equivalent.*

PROOF: The first statement follows directly from Propositions 5.6 and 5.7 by transitivity. It also follows from Proposition 4.4 that $\mathcal{A}/\mathcal{J}_{\min}(\mathcal{A})$ has trivial minimal ideal, while $\mathcal{J}_{\min}(\mathcal{A}/\mathcal{I}) = \mathcal{J}_{\min}(\mathcal{A})/\mathcal{I} \neq 0$. So the result follows by Corollary 5.5. $\qquad\square$

6   Examples from Differential Geometry

In this section, we describe closed *-ideals and minimal ideals of *-algebras arising in differential geometry. We start with a description of the closed *-ideals in $C^\infty(M)$, the algebra of complex-valued smooth functions on an m-dimensional smooth manifold M.

Let us first recall that given a positive Borel measure μ on M, with compact support, we can define a positive linear functional ω_μ on $C^\infty(M)$ by integration with respect to this measure and consider the corresponding GNS representation of $C^\infty(M)$, denoted by π_μ. We observe that in this case, $\ker \pi_\mu = \{f \in C^\infty(M) \text{ with } \int f|g|^2 d\mu = 0 \text{ for all } g \in C^\infty(M)\}$ coincides with the Gel'fand ideal $\{f \in C^\infty(M) \text{ with } \int |f|^2 d\mu = 0\}$. To each closed set $F \subseteq M$, there naturally corresponds a *-ideal of $C^\infty(M)$ given by

$$\mathcal{I}_F = \{f \in C^\infty(M) \text{ such that } f|_F = 0\}, \tag{6.1}$$

called the *vanishing ideal* of F. The following proposition shows that these ideals are exactly the closed ideals of $C^\infty(M)$.

Proposition 6.1 *A *-ideal $\mathcal{I} \subseteq C^\infty(M)$ is closed if and only if it is the vanishing ideal of some closed set $F \subseteq M$. Moreover, $\mathcal{I} = \ker \pi_\omega$, for some GNS representation π_ω, if and only if $\mathcal{I} = \mathcal{I}_K$ for some compact set $K \subseteq M$.*

PROOF: Let $F \neq \emptyset$ be a closed set in M. Denote by $\mathcal{M}_F$ the set of positive compactly supported Borel measures on M with support in F. By the continuity of functions in $C^\infty(M)$, it follows that $\mathcal{I}_F = \bigcap_{\mu \in \mathcal{M}_F} \ker \pi_\mu$. Thus $\mathcal{I}_F$ is closed. Conversely, if $\mathcal{I}$ is closed, then it is the intersection of kernels of GNS representations of

some positive linear functionals. But the positive linear functionals in $C^\infty(M)$ are given by positive Borel measures, see e.g. [9, App. B]. So we can write $\mathfrak{I} = \ker\pi$, for $\pi = \oplus_\alpha \pi_{\mu_\alpha}$ and define $F = \overline{\bigcup_\alpha \operatorname{supp}\mu_\alpha}$. It is then easy to check that $\mathfrak{I} = \mathfrak{I}_F$. The last assertion follows from the fact that if K is compact, then there is a positive Borel measure with support exactly K. $\qquad\square$

We recall that the vanishing ideals of closed sets in $C^\infty(M)$ are exactly the closed ideals with respect to the (weak) C^0-topology induced from $C(M)$, that is, the locally convex topology generated by the family of semi-norms $\rho_K(f) = \sup_{x\in K}|f(x)|$, for $K \subseteq M$ compact (this follows from [35, Thm. 3]). Hence we have

Corollary 6.2 *A *-ideal $\mathfrak{I}$ in $C^\infty(M)$ is closed (in the algebraic sense) if and only if it is closed with respect to the (weak) C^0-topology.*

We remark that similar results hold for the algebra $C_0^\infty(M)$ of compactly supported smooth complex-valued functions on M.

Proposition 6.3 *Let $\mathfrak{I} \subseteq C_0^\infty(M)$ be a *-ideal. Then $\mathfrak{I}$ is closed if and only if $\mathfrak{I}$ is a vanishing ideal if and only if $\mathfrak{I}$ is the kernel of a GNS representation.*

It is clear that the minimal ideal of $C^\infty(M)$ is trivial. We will now consider an example with nontrivial minimal ideal, namely the sections of the Grassmann algebra bundle over M. This generalizes the example of the Grassmann algebra discussed in the introduction. We denote by $\bigwedge^\bullet T^*M$ the complexified Grassmann algebra bundle over M. There are several possibilities to define fibrewise *-involutions for $\bigwedge^\bullet T^*M$. We do it by declaring the real one-forms to be Hermitian and extending this to an antilinear anti-automorphism of the Grassmann algebra bundle. Then $\Gamma^\infty(\bigwedge^\bullet T^*M)$ becomes a *-algebra over $\mathbb{C}$. Denote by $\bigwedge^+ T^*M$ the bundle of forms of positive degree. We have the following characterization of the minimal ideal of $\Gamma^\infty(\bigwedge^\bullet T^*M)$.

Proposition 6.4 *The minimal ideal of $\Gamma^\infty(\bigwedge^\bullet T^*M)$ is given by $\Gamma^\infty(\bigwedge^+ T^*M)$. Thus the representation theories of $\Gamma^\infty(\bigwedge^\bullet T^*M)$ and $C^\infty(M)$ are equivalent but the algebras are not formally Morita equivalent. The closed ideals of $\Gamma^\infty(\bigwedge^\bullet T^*M)$ are of the form $\mathfrak{I}\oplus\Gamma^\infty(\bigwedge^+ T^*M)$, where $\mathfrak{I}$ is a closed ideal of $C^\infty(M)$.*

PROOF: Due to the nilpotence properties of forms with positive degree, it is easy to see that $\Gamma^\infty(\bigwedge^+ T^*M)$ has to be contained in the minimal ideal. On the other hand, $C^\infty(M) = \Gamma^\infty(\bigwedge^0 T^*M)$ has sufficiently many positive linear functionals and thus the first statement follows. The other statement is clear from the above results and Section 5. $\qquad\square$

Let $E \to M$ be a complex Hermitian vector bundle over M, with k-dimensional fibers and fiber metric h. We write $\mathsf{End}(E) \to M$ for the corresponding endomorphism bundle and $\Gamma^\infty(E)$, $\Gamma^\infty(\mathsf{End}(E))$ for the spaces of smooth sections of E and $\mathsf{End}(E)$, respectively. We denote the trivial bundle $M \times \mathbb{C}^k \to M$ by $t(\mathbb{C}^k)$. The final part of this section shows that $\Gamma^\infty(E)$ is a $(\Gamma^\infty(\mathsf{End}(E))$-$C^\infty(M))$-equivalence bimodule and gives a description of the closed *-ideals in $\Gamma^\infty(\mathsf{End}(E))$.

We recall that E can always be regarded as a subbundle of some trivial bundle $t(\mathbb{C}^N)$ over M (the proof in [16, Ch. I, Thm. 6.5] works for arbitrary manifolds since in this case vector bundles are of finite type, see [20, Sect. 5], [21, Lem. 2.7]). Denote $C^\infty(M)$ by $\mathcal{A}$. The natural Hermitian metric in $t(\mathbb{C}^N)$ induces an $\mathcal{A}$-valued inner product on $\Gamma^\infty(t(\mathbb{C}^N)) \cong \mathcal{A}^N$ and the embedding $i : E \hookrightarrow t(\mathbb{C}^N)$ can be assumed to preserve metrics ([16, Ch. I, Thm. 8.8]). Identifying $\Gamma^\infty(E)$ with a submodule of $\mathcal{A}^N$, we can write $\Gamma^\infty(E) \oplus \Gamma^\infty(E)^\perp \cong \mathcal{A}^N$ and hence there is a projection $Q \in M_N(\mathcal{A})$ so that $\Gamma^\infty(E) \cong Q\mathcal{A}^N$ as right $\mathcal{A}$-modules. Note also that $QM_N(\mathcal{A})Q \cong \Gamma^\infty(\mathsf{End}(E))$ as *-algebras. It

is clear that $\Gamma^\infty(E)$ is a $(\Gamma^\infty(\mathsf{End}(E))$-$C^\infty(M))$-bimodule equipped with $\Gamma^\infty(\mathsf{End}(E))$ and $C^\infty(M)$ valued inner products given by $(u,v) \mapsto \Theta_{u,v} \in \Gamma^\infty(\mathsf{End}(E))$, where $\Theta_{u,v}(w) = u \cdot h(v,w)$, and $(u,v) \mapsto h(u,v) \in C^\infty(M)$, for $u,v,w \in \Gamma^\infty(E)$. We finally observe that this bimodule structure on $\Gamma^\infty(E)$ (with the algebra valued inner products) is just the one considered for $Q\mathcal{A}^N$ in [9, Sect. 6] after the aforementioned identifications. Recall that a projection $Q \in M_N(\mathcal{A})$ is called *full* if $\mathbb{C}$-span $\{TQS \ ; \ T,S \in M_N(\mathcal{A})\} = M_N(\mathcal{A})$. The following observation is well-known.

Lemma 6.5 *Let $Q \in M_N(C^\infty(M))$ be a non-zero projection. Then Q is full.*

It then follows from [9, Sect. 6] that $\Gamma^\infty(E)$ satisfies all the properties of a $(\Gamma^\infty(\mathsf{End}(E))$-$C^\infty(M))$-equivalence bimodule, except possibly for property **Q**, which we now show holds in general. Let $(\pi, \mathfrak{H})$ be a *-representation of $QM_N(\mathcal{A})Q \cong \Gamma^\infty(\mathsf{End}(E))$ and $\widetilde{\mathfrak{K}} = \overline{Q\mathcal{A}^N} \otimes_{QM_N(\mathcal{A})Q} \mathfrak{H}$. Note that $\overline{Q\mathcal{A}^N}$ is again just $\Gamma^\infty(E)$ (with left and right actions given by the adjoint of the previous ones). Given $s_i \in \Gamma^\infty(E)$ and $\psi_i \in \mathfrak{H}$, we must show that

$$\left\langle \sum_i s_i \otimes \psi_i, \sum_j s_j \otimes \psi_j \right\rangle_{\widetilde{\mathfrak{K}}} \geq 0, \tag{6.2}$$

where $\langle s_1 \otimes \psi_1, s_2 \otimes \psi_2 \rangle_{\widetilde{\mathfrak{K}}} = \langle \psi_1, \pi(\langle s_1, s_2 \rangle)\psi_2 \rangle_{\mathfrak{H}}$. We then have

Lemma 6.6 $\Gamma^\infty(E)$ *satisfies property* **Q**.

PROOF: Assuming $E = t(\mathbb{C}^k)$, the s_i's are just $\mathbb{C}^k$-valued functions on M and $\mathsf{End}(E) = t(M_k(\mathbb{C}))$. Note that if e_j denote the canonical section basis of E, then for each i one can find a $T_i \in t(M_k(\mathbb{C}))$ so that $e_1 \cdot T_i = s_i$ (recall that $e_1 \cdot T_i$ means $T_i{}^* e_1$, where the last expressions is just the usual matrix multiplication). For instance, define T_i to be the matrix valued function with s_i as the first row and zero everywhere else. Now the proof follows the usual trick: $\left\langle \sum_i s_i \otimes \psi_i, \sum_j s_j \otimes \psi_j \right\rangle_{\widetilde{\mathfrak{K}}} = \left\langle \sum_i e_1 T_i \otimes \psi_i, \sum_j e_1 T_j \otimes \psi_j \right\rangle_{\widetilde{\mathfrak{K}}} = \left\langle e_1 \otimes \sum_i T_i \psi_i, e_1 \otimes \sum_j T_j \psi_j \right\rangle_{\widetilde{\mathfrak{K}}} \geq 0$.

To prove the result in general, let U_k, $k = 1, \ldots, r$ be an open cover of M so that $E|_{U_k}$ is trivial. Let χ_k be a quadratic partition of unity subordinated to this cover. Then observe that for $s_1, s_2 \in \Gamma^\infty(E)$, we have that $\langle s_1, s_2 \rangle_{QM_N(\mathcal{A})Q} = \sum_k \langle \chi_k s_1, \chi_k s_2 \rangle_{QM_N(\mathcal{A})Q}$ (since $\sum_k \langle \chi_k s_1, \chi_k s_2 \rangle(x) = \sum_k \Theta_{\chi_k s_1(x), \chi_k s_2(x)} = \sum_k \chi_k(x)\overline{\chi_k}(x)\Theta_{s_1(x),s_2(x)} = \langle s_1, s_2 \rangle(x))$. Hence,

$$\left\langle \sum_i s_i \otimes \psi_i, \sum_i s_i \otimes \psi_i \right\rangle_{\widetilde{\mathfrak{K}}} = \sum_{i,j} \langle \psi_i, \langle s_i, s_j \rangle \psi_j \rangle_{\widetilde{\mathfrak{K}}} = \sum_{i,j,k} \langle \psi_i, \langle \chi_k s_i, \chi_k s_j \rangle \psi_j \rangle_{\widetilde{\mathfrak{K}}}.$$

Now note that the last expression is just $\sum_k \langle \sum_i (\chi_k s_i) \otimes \psi_i, \sum_i (\chi_k s_i) \otimes \psi_i \rangle$. But since $\mathrm{supp}\, \chi_k s_i \subseteq U_k$, and $E|_{U_k}$ is trivial, the result follows from the previous discussion. $\qquad \square$

Proposition 6.7 *If E is a complex hermitian vector bundle over M, then $\Gamma^\infty(E)$ is a $(C^\infty(M)$-$\Gamma^\infty(\mathsf{End}(E)))$-equivalence bimodule.*

One can also consider compactly supported functions and sections. In this case, $C_0^\infty(M)$ and $\Gamma_0^\infty(\mathsf{End}(E))$ are no longer unital in general, but have approximate identities. So the results of [9, Sect. 6] can still be applied and the same proof used for Prop. 6.7 shows that

Proposition 6.8 $\Gamma_0^\infty(E)$ *is a $(C_0^\infty(M)$-$\Gamma_0^\infty(\mathsf{End}(E)))$-equivalence bimodule.*

We finally remark that combining the results of Section 5, one easily arrives at a characterization of the closed *-ideals of $\Gamma^\infty(\mathsf{End}(E))$ (and analogously for $\Gamma_0^\infty(\mathsf{End}(E))$).

Corollary 6.9 *A *-ideal $\mathfrak{I} \subseteq \Gamma^\infty(\mathsf{End}(E))$ is closed if and only it is a vanishing ideal, i.e. of the form $\{B \in \Gamma^\infty(\mathsf{End}(E)) \mid B(x) = 0 \ \forall x \in F\}$ for some closed $F \subseteq M$.*

Acknowledgements

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